Chordal Löwner Evolution

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$\mathbb{H} = \{\operatorname{Im} z > 0\}$ - upper half-plane.

Def. $A \subset \mathbb{H}$ - compact hull if 1) $\overline{A}$ - compact,
2) $A = \overline{A} \cap \mathbb{H}$.
3) $\mathbb{H} \setminus A$ - simply-connected domain.

Main example Curve γ from 0 to ∞ generates family of hulls.

Fact. $\exists$ unique map $g_A : \mathbb{H} \setminus A \to \mathbb{H}$: $\lim_{z \to \infty} (g_A(z) - z) = 0$ (Hydrodynamic normalization).

Pf By Riemann, $\exists g : \mathbb{H} \setminus A \to \mathbb{H}$, with $g(\infty) = \infty$. Extendable to $\tilde{g} : \hat{\mathbb{C}} \setminus (\text{los}(A \cup \bar{A}))^c \to \hat{\mathbb{C}}$, mapping $\mathbb{R} \to \mathbb{R}$. & expansion at ∞, $\tilde{g}(z) = Az + B + \ldots$, where $A, B \in \mathbb{R}$.
Normalize ∎

Def. (Half-plane capacity)
$$\operatorname{hcap}(A) = \lim_{z \to \infty} z(g_A(z) - z).$$

Lemma If $r > 0$, $\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A)$.

Pf $g_{rA}(z) = r\, g_A(\frac{z}{r})$ ∎

Examples 1) $A = \mathbb{D} \cap \mathbb{H}$, $g_A(z) = z + \frac{1}{z}$, $\operatorname{hcap} A = 1$.

2) $A = [0, i]$, then $g_A(z) = \sqrt{z^2+1} = z + \frac{1}{2z} + \ldots \Rightarrow \operatorname{hcap} A = \frac{1}{2}$.

Lemma $\operatorname{hcap} A \geq 0$.

Pf $V(z) := \operatorname{Im}(z - g_A(z))$.
Then $\lim_{z \to \infty} V(z) = 0$, $V(z) \geq 0$ on $\mathbb{R}$. By maximum principle, $V(z) > 0$ on $\mathbb{H}$.
$\operatorname{hcap} A = \lim_{z \to \infty} z(z - g_A z) = -\lim_{y \to \infty} iy\, iV(iy) \geq 0$ ∎

Let A - locally connected, $f_A := g_A^{-1}$,
I - minimal interval containing $g_A(\overline{A} \cap \mathbb{R})$. Extend f to I by local connectivity.
Extend to f^* on $\mathbb{C} \setminus \mathbb{R}$, by reflection.

By Cauchy, $f^*(w) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f^*(z)}{z - w}\, dz + \frac{1}{2\pi i} \int_I \frac{f_A(x) - \overline{f_A(x)}}{x - w}\, dx$, if $R > |w|$

$\overline{}$ $2 I m f_A(x)$

Since at ∞, $f^*(z) = z - \operatorname{hcap} A \cdot \frac{1}{z} + \ldots$
we get
$$\lim_{R \to \infty} \int_{|z|=R} \frac{f^*(z)}{z - w}\, dz = \lim_{R \to \infty} \int_{|z|=R} \frac{z}{z - w}\, dz = 2\pi i w.$$

Thus $f^*(w) - w = \frac{1}{\pi} \int_I \frac{\operatorname{Im} f_A(x)}{x - w}\, dx$. Multiply by w, take $w \to \infty$,
get
$$\boxed{\operatorname{hcap} A = \frac{1}{\pi} \int_I \operatorname{Im} f_A(x)\, dx}$$

In particular, if $A \neq \emptyset$, $\operatorname{hcap} A > 0$; if A is locally connected.

Lemma. $A \subset A'$ - compact hulls. Then
$$\operatorname{hcap} A' = \operatorname{hcap} A + \operatorname{hcap} g_A(A' \setminus A).$$
In particular, $\operatorname{hcap} A' \geq \operatorname{hcap} A$.

Pf. $g_{A'} = g_{g_A(A' \setminus A)} \circ g_A$, extend at ∞ ∎

Lemma. If $A \neq \emptyset$ - compact hull $\Rightarrow \operatorname{hcap} A > 0$.

Pf. $\exists A' \subset A$ - locally connected, $\neq \emptyset$ ∎

Caratheodory convergence: $A \cup \bar{A}$ converge in the usual Caratheodory sense w.r.t. ∞.

Class P: ...

Class $\mathcal{P}$: $\operatorname{Re} p > 0$, $\lim zp(z) = 1$.

Herglotz representation: $p(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}$, $\operatorname{supp}\mu = \overline{\{t : \lim_{y\to 0+} p(t+iy) > 0\}}$.

Chordal L.C.: $f_t : \mathbb{H} \to \Omega_t := \mathbb{H}\setminus K_t$, where K_t – continuous growing family of compact hulls. Normalized L.C.: $\operatorname{hcap} K_t = 2t$.

Löwner equations:

$$\frac{\partial f_t}{\partial t} = -2f_t'(z)\int_{\mathbb{R}} \frac{d\mu_t(x)}{z-x}, \quad f_0(z) \equiv z. \qquad g_t := f_t^{-1}.$$

$$\frac{\partial g_t(z)}{\partial t} = 2\int \frac{d\mu_t(x)}{g_t(z)-x}, \quad g_0(z) \equiv z.$$

For curves:

$$\frac{\partial g_t}{\partial t}(z) = \frac{2}{g_t(z)-\lambda(t)}, \quad \lambda(t) = g_t(\gamma(t)) \text{ – continuous driving function}$$

An analogue of Pommerenke holds, with crosscut separating two maps.

Scaling property: (Brownian scaling)

Driving function for αK_t is $\alpha^{-1}\lambda(\alpha^2 t)$ (since $g_{\alpha K}(z) = \alpha g_K(\frac{z}{\alpha})$).

Corollary: $\alpha\sqrt{t}$ generates straight line segment.